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A COUNTEREXAMPLE ON NUMERICAL RADIUS ATTAINING OPERATORS

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ABSTRACT

We answer a question posed by B. Sims in 1972, by exhibiting an example of a Banach space X such that the numerical radius attaining operators on X are not dense. Actually, X is an old example used by J. Lindenstrauss to solve the analogous problem for norm attaining operators, but the proof for the numerical radius seems to be much more difficult. Our result was conjectured by C. Cardassi in 1985.

Introduction

Recall the definition of the **numerical range** $V(T)$ of a bounded linear operator T on a Banach space X. It is given by

$$V(T) = \{x^*(Tx) : x \in X, x^* \in X^*, \|x^*\| = \|x\| = x^*(x) = 1\}$$

where X^* denotes the dual space of X. This is an old notion going back to O. Toeplitz [21], who defined in 1918 the numerical range of an operator on the euclidean n-space, his definition being meaningful for operators on an arbitrary Hilbert space. General information on the numerical range of an operator on Hilbert space can be found in the book by P. Halmos [15]. The extension to Banach space operators was done in 1961–62 by G. Lumer [17] and F. Bauer [4]. A systematic discussion of numerical ranges of operators on Banach spaces, emphasizing the connections with spectral theory of operators and general Banach algebra theory, can be found in the books by F. Bonsall and J. Duncan [7,8]. For

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more condensed information the reader is referred to the survey article by the same authors [9].

The **numerical radius** of T is defined by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$$

and v is clearly a seminorm on the operator space $L(X)$ satisfying that $v(T) \leq \|T\|$ for all T. It is said that T **attains its numerical radius** when the supremum defining $v(T)$ is actually a maximum and we will denote by $R(X)$ the set of numerical radius attaining operators on X. If X is finite dimensional, one has clearly that the numerical range of any operator on X is compact, so $R(X) = L(X)$ in this case. Even in separable Hilbert space it is easy to find a diagonal self-adjoint operator which does not attain its numerical radius. In his dissertation [20], B. Sims proved that every self-adjoint operator on a Hilbert space can be approximated in norm by numerical radius attaining self-adjoint operators, and he posed the general problem if $R(X)$ is norm-dense in $L(X)$ for any Banach space X. Partial affirmative answers to this question have been obtained by I. Berg and B. Sims [5], C. Cardassi [10,11,12,13] and M. Acosta and the author [2]. The fact that $R(X)$ is dense in $L(X)$ when X has the Radon–Nikodym property [3], which includes the results in [2,5,12] seems to be the most relevant in this direction.

The purpose of this paper is answering the general question to the negative, by exhibiting a Banach space X such that $R(X)$ is not dense in $L(X)$. Actually we show that the Banach space used by J. Lindenstrauss in [16] to show that the norm attaining operators from a Banach space into itself need not be dense, also does the job for the numerical radius. More concretely, let Y denote the space c_0 provided with the strictly convex norm $|\cdot|$ given by

$$|y| = \|y\| + \left(\sum_{n=1}^{\infty} \frac{|y(n)|^2}{2^n}\right)^{1/2}$$

where $\|\cdot\|$ is the usual norm on c_0 and $y(n)$ is the nth term of the sequence $y \in Y$. Now consider the space $X = Y \oplus_\infty c_0$, where c_0 carries its usual norm and we put the maximum norm on the direct sum. We will prove that $R(X)$ is not dense in $L(X)$. This result was conjectured by C. Cardassi in [10]. Our proof is rather technical and it makes essential use of some ideas concerning radial derivatives of the norm which are always in close connection with numerical range problems. However, as far as we know, these techniques are applied for the first time to deal with numerical radius attaining operators.

The strong parallelism between norm attaining and numerical radius attaining operators can not be ignored. It clearly arises from the above mentioned partial answers to Sims problem, but the present counterexample points in the same direction. Let us remark some similarities and differences between both kinds of problems. Note that an operator T attains its norm if and only if there are $x_0 \in X, x_0^* \in X^*$, with $\|x_0\| = \|x_0^*\| = 1$ such that $|x_0^*(Tx_0)| = \|T\|$, so we have the problem of maximizing a certain function on the product of the unit spheres of X and X^*. When dealing with the numerical radius, the same optimization problem is constrained by the condition $x_0^*(x_0) = 1$ and the function to be maximized is only considered on the fairly more complicated set

$$\Pi(X) = \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = x^*(x) = 1\}$$

Only for very special spaces X the set $\Pi(X)$ is well known and this is probably the reason why the arguments dealing with the numerical radius are usually more involved than the corresponding arguments for the norm.

If an operator T satisfies $v(T) = \|T\|$ and T attains its numerical radius, then it clearly attains its norm as well. However, it is not difficult to give an example of an operator T on the space l_2 such that $v(T) = \|T\|$, T attains its norm but not its numerical radius. On the other hand, if $v(T) < \|T\|$, then T may attain its numerical radius while not its norm. For these and related examples showing that both optimization problems are independent we refer to [1]. Let us finally remark the main difference between both problems. The numerical range only makes sense for operators from a Banach space into itself. In particular we can not imagine what could be the numerical radius counterpart of the celebrated properties A and B, introduced by J. Lindenstrauss in [16].

1. Preliminary results

The closed unit ball and the unit sphere of a (real or complex) Banach space X will be denoted by B_X and S_X respectively, X^* will be the dual space and $L(X)$ the Banach space of bounded linear operators on X. For $T \in L(X)$, T^* will be the adjoint operator. Given $u \in S_X$, the set of normalized support functionals for B_X at u will be denoted by $D(X,u)$, or simply $D(u)$ if X is clear from the context. Thus,

$$D(u) = \{x^* \in S_{X^*} : x^*(u) = 1\}$$

This is a nonempty, convex and w^*-compact set and, for $x \in X$ we will write

$$\tau(u,x) = \max\{\text{Re } x^*(x) : x^* \in D(u)\}$$

It is an old result, due to S. Mazur (see [18] or [14; Theorem V.9.5]), that $\tau(u,x)$ is the right derivative of the norm at the point u in the direction of x, that is

$$\tau(u,x) = \lim_{t \to 0+} \frac{\|u + tx\| - 1}{t}$$

As a function of x, τ is subadditive,

$$\tau(u, x_1 + x_2) \leq \tau(x_1) + \tau(x_2) \qquad (x_1, x_2 \in X)$$

and it follows that τ is also continuous in the second variable, in fact

$$|\tau(x_1) - \tau(x_2)| \leq \|x_1 - x_2\| \quad (x_1, x_2 \in X)$$

We will use the following easy extension of Mazur's result, a "chain rule" involving norm derivatives.

LEMMA 1.1 (19; Lemma 1.6): *Let F be a function defined on the real interval $[0,\delta]$ with values in the Banach space X. Assume that $F(0) \neq 0$ and that F is differentiable at the origin. Then the real function G defined by*

$$G(t) = \|F(t)\| \qquad (0 \leq t \leq \delta)$$

is differentiable at the origin and

$$G'(0) = \tau\left(\frac{F(0)}{\|F(0)\|}\ ,\ F'(0)\right).$$

The numerical range of an operator $T \in L(X)$ is given by

$$V(T) = \{x^*(Tx) : (x, x^*) \in \Pi(X)\}$$

where

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^* \in D(x)\}$$

The numerical radius of T is then given by

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$$

and we say that T attains its numerical radius if $|x_0^*(Tx_0)| = v(T)$ for some $(x_0, x_0^*) \in \Pi(X)$. $R(X)$ will be the set of numerical radius attaining operators on X.

The Banach space X we are interested in has the form $X = Y \oplus_\infty Z$ where Y, Z are Banach spaces and we use the symbol $\oplus_\infty$ to indicate that we take the maximum norm on the direct sum,

$$\|y + z\| = \max\{\|y\|, \|z\|\} \quad (y \in Y, z \in Z)$$

Several easy facts concerning numerical ranges of operators on such a space X are collected together in the following statement.

LEMMA 1.2: *Let Y, Z be Banach spaces, $X = Y \oplus_\infty Z$ and P, Q the projections from X onto Y, Z respectively. For $T \in L(X)$ we have*

i) $v(T) = \max\{v(PT), v(QT)\}$.

ii) If $T \in R(X)$ and $v(PT) > v(QT)$, then $PT \in R(X)$.

iii) $v(PT) = \sup\{|y^*(PT(y+z))| : (y, y^*) \in \Pi(Y), z \in B_Z\}$ and $PT \in R(X)$ if and only if this supremum is attained.

Proof: Note that the adjoint projections P^* and Q^* are complementary L-projections, that is,

$$\|x^*\| = \|P^*x^*\| + \|Q^*x^*\| \quad (x^* \in X^*)$$

It follows easily that

$$D(x) = \mathrm{co}\,([D(x) \cap P^*(X^*)] \cup [D(x) \cap Q^*(X^*)])$$

for all x in S_X, where "co" denotes convex hull.

To prove i), fix $(x, x^*) \in \Pi(X)$ and write $x^* = (1-r)u^* + rv^*$ with $0 \le r \le 1$, $u^*, v^* \in D(x)$, $P^*(u^*) = u^*$, $Q^*(v^*) = v^*$. If either $D(x) \cap P^*(X^*) = \emptyset$ or $D(x) \cap Q^*(X^*) = \emptyset$, we simply take $r = 1$ or $r = 0$. In any case we have

$$|x^*(Tx)| = |(1-r)u^*(PTx) + rv^*(QTx)| \le$$

$$\le (1-r)v(PT) + rv(QT) \le \max\{v(PT), v(QT)\}$$

and we have shown that

$$v(T) \le \max\{v(PT), v(QT)\}.$$

To see, for example, that $v(PT) \le v(T)$, assume that $|x^*(PTx)| > 0$ with $(x, x^*) \in \Pi(X)$. Then $0 < \|P^*x^*\| \le 1$ and $\frac{P^*x^*}{\|P^*x^*\|} \in D(x)$, so

$$|x^*(PTx)| = |[P^*x^*](Tx)| \le \left| \frac{P^*x^*}{\|P^*x^*\|}(Tx) \right| \le v(T),$$

and $v(PT) \le v(T)$, as required.

To prove ii), let $(x, x^*) \in \Pi(X)$ be such that

$$|x^*(Tx)| = v(T) = v(PT) > v(QT).$$

Since $|x^*(QTx)| \le v(QT) < |x^*(Tx)|$, we have $x^*(PTx) \ne 0$, so $P^*x^* \ne 0$. We claim that $P^*x^* = x^*$. Otherwise we would have $u^* := \frac{P^*x^*}{\|P^*x^*\|} \in D(x)$ and $v^* := \frac{Q^*x^*}{\|Q^*x^*\|} \in D(x)$, so

$$|x^*(Tx)| = \|P^*x^*\| \; |u^*(PTx)| + \|Q^*x^*\| \; |v^*(QTx)| \le$$

$$\le \|P^*x^*\| v(PT) + \|Q^*x^*\| v(QT) < v(T),$$

a contradiction. Thus our claim is proved, so

$$v(PT) = |x^*(Tx)| = |x^*(PTx)|$$

and we have shown that PT attains its numerical radius.

Let us finally show that iii) holds. If $(y, y^*) \in \Pi(Y)$ and $z \in B_Z$, it is plain that $y^*(PT(y+z))$ belongs to the numerical range of PT (just take $x = y + z \in S_X$ and consider y^* as an element of X^* vanishing on Z, then $(x, y^*) \in \Pi(X)$), so only one inequality requires a proof. For fixed $(x, x^*) \in \Pi(X)$ we must find $(y, y^*) \in \Pi(Y)$, $z \in B_Z$ such that

$$|x^*(PTx)| \leq |y^*(PT(y+z))|$$

and we can clearly assume that $P^*x^* \neq 0$. Then

$$1 = \frac{P^*x^*}{\|P^*x^*\|}(x) \leq [P^*x^*](x) = x^*(Px) \leq \|Px\| \leq \|x\| = 1,$$

so we have $(y, y^*) \in \Pi(Y)$ where $y = Px$ and y^* is (the restriction to Y of) $\frac{P^*x^*}{\|P^*x^*\|}$. By taking $z = Qx$ we get

$$|x^*(PTx)| = |[P^*x^*](PTx)| \leq |y^*(PTx)| = |y^*(PT(y+z))|,$$

as required. ■

Two direct consequences of the above lemma will be useful for us. First, if X is as in the lemma and an operator $T \neq 0$ satisfying $T = PT$ is the limit in norm of a sequence $\{T_n\}$ of operators in $R(X)$, then the sequence $\{PT_n\}$ also converges to T while $\{QT_n\}$ converges to zero, so we will have

$$v(T_n) = v(PT_n) > v(QT_n) \qquad \text{and} \qquad PT_n \in R(X)$$

for large enough n, and we can assume that $PT_n = T_n$ for all n, from the very beginning. Second, the numerical radius of operators T satisfying $PT = T$ only involves the set $\Pi(Y)$, which is simpler than $\Pi(X)$. We now give an easy description of $\Pi(Y)$ for the kind of space Y we are interested in.

LEMMA 1.3: *Let $(Y_0, \|\cdot\|)$, $(H, \|\cdot\|)$ be Banach spaces and W a one-to-one bounded linear operator from Y_0 into H. Define an equivalent norm $|\cdot|$ on Y_0 by*

$$|y| = \|y\| + \|Wy\|$$

and let Y denote the new Banach space obtained in this way. Then

i) The dual norm $|\cdot|$ on Y^* is given by

$$|y^*| = \min\{\max\{\|y^* - W^*h^*\|, \|h^*\|\} : h^* \in H^*\},$$

for all y^* in Y^*.

ii) $$D(Y,y) = D\left(Y_0, \frac{y}{\|y\|}\right) + W^*\left[D\left(H, \frac{Wy}{\|Wy\|}\right)\right],$$
for all y in S_Y.

Proof: i) This follows from a standard dualization argument. The mapping $\Phi : Y \to Y_0 \times H$ given by

$$\Phi(y) = (y, Wy)$$

is an isometric embedding when $Y_0 \times H$ is normed by

$$\|(y,h)\| = \|y\| + \|h\| \quad (y \in Y_0, h \in H),$$

so the mapping $u + \Phi(Y)^\circ \longrightarrow \Phi^*(u)$ is an isometric isomorphism from $(Y_0 \times H)^*/\Phi(Y)^\circ$ onto Y^*. Since

$$\Phi^*(y^*, h^*) = y^* + W^*h^* \quad (y^* \in Y_0^*, h^* \in H^*),$$

we have

$$|y^*| = \|(y^*, 0) + \Phi(Y)^\circ\|,$$

and the result follows by using that $(Y_0 \times H)^* \cong Y_0^* \oplus_\infty H^*$ and

$$\Phi(Y)^\circ = \ker \Phi^* = \{(-W^*h^*, h^*) : h^* \in H^*\}.$$

Note that the infimum defining the quotient norm is attained, for $\Phi(Y)^\circ$ is a w^*-closed subspace.

ii) Given $y \in S_Y$ and $y^* \in D(y)$, let $h^* \in H^*$ be such that

$$\max\{\|y^* - W^*h^*\|, \|h^*\|\} = 1$$

Then we have

$$1 = \text{Re } y^*(y) = \|y\|\text{Re } [y^* - W^*h^*]\left(\frac{y}{\|y\|}\right) + \|Wy\|\text{Re } h^*\left(\frac{Wy}{\|Wy\|}\right) \le$$

$$\le \|y\| \; \|y^* - W^*h^*\| + \|Wy\| \; \|h^*\| \le$$

$$\le (\|y\| + \|Wy\|) \; \max\{\|y^* - W^*h^*\|, \|h^*\|\} = 1.$$

It follows that $y^* - W^*h^* \in D\left(Y_0, \frac{y}{\|y\|}\right)$ and $h^* \in D\left(H, \frac{Wy}{\|Wy\|}\right)$.

Conversely, if $y \in S_Y$ and $y^* = u^* + W^*h^*$ with

$$u^* \in D\left(Y_0, \frac{y}{\|y\|}\right) \quad \text{and} \quad h^* \in D\left(H, \frac{Wy}{\|Wy\|}\right),$$

we have

$$|y^*| \leq \max\{\|y^* - W^*h^*\|, \|h^*\|\} = 1$$

and

$$y^*(y) = \|y\|u^*\left(\frac{y}{\|y\|}\right) + \|Wy\|h^*\left(\frac{Wy}{\|Wy\|}\right) = 1,$$

so $|y^*| = 1$ and $y^* \in D(Y, y)$. ■

2. The counterexample

Notation 2.1: Consider the diagonal operator W from c_0 into l_2 satisfying

$$We_n = \rho_n \, h_n \tag{1}$$

where $\{e_n\}$ and $\{h_n\}$ are the unit vector bases of c_0 and l_2 respectively, and $\{\rho_n\}$ is a fixed sequence of positive numbers in l_2. W is clearly a one-to-one bounded linear operator. We will denote by Y the space c_0 when provided with the norm $|\cdot|$ defined by

$$|y| = \|y\| + \|Wy\| =$$

$$= \max\{|y(n)| : n \in \mathbb{N}\} + \left(\sum_{n=1}^{\infty} \rho_n^2 \; |y(n)|^2\right)^{1/2}. \tag{2}$$

Now we consider the space $X = Y \oplus_\infty c_0$, whose norm is defined by

$$\|y + z\| = \max\{|y|, \|z\|\} \quad (y \in Y, z \in c_0) \tag{3}$$

We want to prove that $R(X)$ is not dense in $L(X)$. To this end we consider operators $A \in L(Y)$, $B \in L(c_0, Y)$ and define $T \in L(X)$ by

$$T(y + z) = Ay + Bz \qquad (y \in Y, z \in c_0). \tag{4}$$

Note that this is the general form of an operator on X satisfying that $T(X) \subseteq Y$, equivalently $PT = T$ where P is the projection from X onto Y.

Our proof will easily follow from a satisfactory answer to the following question. If the operator T given by (4) attains its numerical radius, what can be said about A and B?

To motivate our arguments, assume that T attains its norm, so that there are vectors $y_0 \in B_Y$, $z_0 \in B_{c_0}$, such that

$$|Ay + Bz| \leq |Ay_0 + Bz_0|$$

for all $y \in B_Y$, $z \in B_{c_0}$. For large enough n we have $\|z_0 \pm \frac{1}{2}e_n\| \leq 1$, hence

$$|Ay_0 + Bz_0 \pm \frac{1}{2}Be_n| \leq |Ay_0 + Bz_0|.$$

The strict convexity of Y implies that $Be_n = 0$, so B is a finite rank operator. Now, if we only assume that T is the limit in norm of a sequence $\{T_n\}$ of norm attaining operators, since $PT = T$ we can arrange that $PT_n = T_n$ for all n, so the above argument applies to each T_n and this time we get that T is compact. With minor changes this was the argument used by J. Lindenstrauss in [16; Propositions 4 and 5] to show that the set of norm attaining operators on X is not dense.

What happens if we instead assume that T (always given by (4)) attains its numerical radius? Well, in view of Lemma 1.2, there are $(y_0, y_0^*) \in \Pi(Y)$, $z_0 \in B_{c_0}$ such that

$$|y^*(Ay + Bz)| \leq |y_0^*(Ay_0 + Bz_0)| \tag{5}$$

for all $(y, y^*) \in \Pi(Y)$, $z \in B_{c_0}$. As in the previous argument we find a natural number p such that $\|z_0 \pm \frac{1}{2}e_n\| \leq 1$ for $n \geq p$ and we get

$$|y_0^*(Ay_0 + Bz_0 \pm \frac{1}{2}Be_n)| \leq |y_0^*(Ay_0 + Bz_0)|.$$

The scalar field is certainly strictly convex, so we have

$$y_0^*(Be_n) = 0 \quad \text{for } n \geq p,$$

in other words, the sequence $B^*y_0^* \in l_1$ has only finitely many nonzero terms. This seems to be a very poor information, note that we intend to put the terms of a convergent sequence in place of T and try to get something about the limit operator, but the pair (y_0, y_0^*) as well as z_0 (hence also p) depend on T and will change with T, probably without control. Nevertheless, this small piece of information will be crucial to our proof, so it is worth pointing it out.

FACT 2.2: If the operator T defined by (4) attains its numerical radius and $(y_0, y_0^*) \in \Pi(Y)$, $z_0 \in B_{c_0}$ are such that

$$|y_0^*(Ay_0 + Bz_0)| = v(T),$$

then

$$B^* y_0^*(e_n) = 0$$

for large enough n.

Let us go back to (5) in order to get some further information. For fixed $(y, y^*) \in \Pi(Y)$, by rotating z we can arrange that

$$|y^*(Ay + Bz)| = |y^*(Ay)| + |y^*(Bz)|,$$

so we actually have

$$|y^*(Ay)| + |y^*(Bz)| \leq |y_0^*(Ay_0 + Bz_0)|,$$

and this still holds for all z in B_{c_0}. By taking the supremum over z we get

$$|y^*(Ay)| + \|B^* y^*\| \leq |y_0^*(Ay_0 + Bz_0)|,$$

and this is our second piece of information.

FACT 2.3: Under the assumptions of Fact 2.2, we have

$$|y^*(Ay)| + \|B^* y^*\| \leq |y_0^*(Ay_0)| + \|B^* y_0^*\| \tag{6}$$

for all (y, y^*) in $\Pi(Y)$.

Let us explain in advance how the above two facts will entrain severe restrictions on the operators A and B. More concretely, if the norm of A is small, then B cannot be close to the formal identity from c_0 into Y. Why is this so? We know that $B^* y_0^*$ is a quasi-null sequence in l_1, hence if we move a bit from y_0^* in the direction of e_n^* with large enough n ($\{e_n^*\}$ is the sequence of biorthogonal functionals to the basis $\{e_n\}$), so that we get the point $y^* = y_0^* + \varepsilon e_n^*$ (ε small), and B is close to the identity ($B^* e_n^*$ is close to e_n^*), then $\|B^* y^*\|$ will increase (roughly) ε. In view of (6) this increasement has to be compensated by a decreasement of $|y^*(Ay)|$, but it can be expected that also y moves in the direction of e_n, so that the decreasement of $|y^*(Ay)|$ should be of the order $\varepsilon(|y_0^*(Ae_n)| + |e_n^*(Ay_0)|)$

which is much smaller than ε when n is large, for $\{e_n\}$ is a weakly null sequence and $\{e_n^*\}$ is w^*-null. To measure the rate of increasement of $\|B^*(\cdot)\|$ we use radial derivatives of the l_1-norm, so we need $B^*y_0^* \neq 0$ but this condition holds when the norm of A is small enough. To carry over this kind of argument in a serious and successful way, several obstacles have to be overcome and checking all the details will be quite cumbersome. Unfortunately we cannot move from y_0^* in an arbitrary direction while staying in $\Pi(Y)$, what we can do is moving from y_0 in the suitable direction and try to arrange things so that y^* moves in the right way. It is already time to start with our homework.

Assume that the operator T defined in (4) attains its numerical radius and fix a pair $(y_0, y_0^*) \in \Pi(Y)$ satisfying (6). Let us write $r = \|Wy_0\|$, $\|y_0\| = 1 - r$, and consider the set

$$\Lambda = \{n \in \mathbb{N} : |y_0(n)| = 1 - r\} \tag{7}$$

This is a finite set and we have

$$\mu := \max\{|y_0(n)| : n \notin \Lambda\} < 1 - r.$$

Now we fix an element $e \in Y$ such that

$$e(n) = 0 \qquad \text{for } \ n \in \Lambda, \tag{8}$$

choose $\delta > 0$ with $\mu + \delta\|e\| < 1 - r$ and write

$$\phi(t) = |y_0 + te| \ \ , \ \ y_t = \frac{y_0 + te}{\phi(t)} \in S_Y,$$

for $0 \leq t \leq \delta$.

If $n \in \Lambda$ we have

$$|y_t(n)| = \phi(t)^{-1} \ |y_0(n)| = (1 - r) \ \phi(t)^{-1}$$

while if $n \notin \Lambda$,

$$|y_t(n)| \leq \phi(t)^{-1} \ (\mu + \delta\|e\|) < (1 - r) \ \phi(t)^{-1}.$$

It follows that $\|y_t\| = (1-r)\phi(t)^{-1}$, $\|Wy_t\| = \phi(t)^{-1}(\phi(t) - 1 + r)$ and $|y_t(n)| = \|y_t\|$ if and only if $n \in \Lambda$. Thus, y_t is a small perturbation of y_0 which attains its norm $\|\cdot\|$ at the same coordinates as y_0, the variation of $\|y_t\|$ and $\|Wy_t\|$ being controlled by the function $\phi(t)$.

Now we use Lemma 1.3 to see how y_0^* looks like, hence how it should be perturbed to get elements y_t^* such that $(y_t, y_t^*) \in \Pi(Y)$ for $0 \leq t \leq \delta$. Hopefully this perturbation will be smooth enough as a function of t. For $h \in l_2$ we will denote by h^* the functional $(\cdot|h)$, where $(\cdot|\cdot)$ is the inner product on l_2. In particular $\{h_n^*\}$ is the sequence of biorthogonal functionals to the unit vector basis $\{h_n\}$. Note that $h \to h^*$ is the canonical (conjugate linear in the complex case) identification of l_2 with its dual. Also $D(H, h) = \{h^*\}$ for $h \in S_{l_2}$. By Lemma 1.3, y_0^* has the form

$$y_0^* = u_0^* + W^* h_0^* \quad \text{where} \quad u_0^* \in D\left(c_0, \frac{y_0}{1-r}\right) \quad \text{and} \quad h_0 = \frac{1}{r} W y_0.$$

A quick look at the support functionals for the unit ball of c_0 in its usual norm and we realize that u_0^* must have the form

$$u_0^* = \sum_{n \in \Lambda} \eta_n \frac{|y_0(n)|}{y_0(n)} e_n^* \tag{9}$$

where $\eta_n \geq 0$ for $n \in \Lambda$ and $\sum_{n \in \Lambda} \eta_n = 1$. Since

$$\frac{|y_t(n)|}{y_t(n)} = \frac{|y_0(n)|}{y_0(n)}$$

for all $n \in \Lambda$, we have the nice fact that u_0^* also works for y_t (this was the reason to impose on e the restriction (8)), hence we can define

$$y_t^* = u_0^* + W^* h_t^* \quad \text{where} \quad h_t = \frac{W y_t}{\|W y_t\|}$$

and we have $(y_t, y_t^*) \in \Pi(Y)$ for $0 \leq t \leq \delta$. Note that

$$h_t = \frac{W(y_0 + te)}{\phi(t) - 1 + r} = \frac{r h_0 + t h}{\phi(t) - 1 + r} \quad \text{where} \quad h = We,$$

so we can write

$$y_t^* = y_0^* + t\, W^* k_t^* \tag{10}$$

where

$$k_t = t^{-1}(h_t - h_0) = (\phi(t) - 1 + r)^{-1}\left(h - \frac{\phi(t)-1}{t}h_0\right)$$

for $0 < t \leq \delta$ and, what is most important

$$\lim_{t\to 0+} k_t = r^{-1}(h - \phi'(0)\ h_0) =: k_0 \tag{11}$$

in the norm topology of l_2 (actually the k_t's live in the two-dimensional subspace of l_2 generated by h and h_0).

The computation of $\phi'(0)$ is not difficult,

$$\phi'(0) = \lim_{t\to 0+} \frac{|y_0 + te| - 1}{t} = \tau(y_0, e) =$$

$$= \max\ \mathrm{Re}\ \{[u_0^* + W^* h_0^*](e) : u_0^* \in D\left(c_0, \frac{y_0}{1-r}\right)\} = \mathrm{Re}\ W^* h_0^*(e),$$

where, for the last equality, we have used the general form (9) of an element in $D(c_0, \frac{y_0}{1-r})$ and, once more, the restrictions on e (8). Thus, we have

$$\phi'(0) = \mathrm{Re}\ (h|h_0) \tag{12}$$

and (11) reads

$$k_0 = r^{-1}[h - \mathrm{Re}\ (h|h_0)\ \ h_0]. \tag{13}$$

Fortunately, y_t^* is a perturbation of y_0^* which is a smooth function of t, as shown by (10) and (11).

It is time to compute the variations of $|y^*(Ay)|$ and $\|B^* y^*\|$ when we move from the pair (y_0, y_0^*) to (y_t, y_t^*). For the first one we have

$$\phi(t)\ y_t^*(Ay_t) = y_t^*[A(y_0 + te)] =$$

$$= y_0^*(Ay_0) + t\ [W^* k_t^*](Ay_0) + \tag{14}$$

$$+t\ y_0^*(Ae) + t^2\ [W^* k_t^*](Ae).$$

Consider the scalar valued function $F_1(t)$ defined for $0 \leq t \leq \delta$ by

$$F_1(t) = y_t^*(Ay_t).$$

In view of (14), (11) and (12), F_1 is differentiable at the origin with

$$F_1'(0) = [W^* k_0^*](Ay_0) + y_0^*(Ae) - \text{ Re } (h|h_0)\ y_0^*(Ay_0) \tag{15}$$

On the other hand we have the function $F_2 : [0, \delta] \to l_1$ defined by

$$F_2(t) = B^* y_t^* = B^* y_0^* + t\ B^* W^* k_t^*,$$

which is also differentiable at the origin, with

$$F_2'(0) = B^* W^* k_0^*. \tag{16}$$

By Fact 2.3, we have

$$|F_1(t)| + \|F_2(t)\| \leq |F_1(0)| + \|F_2(0)\| \quad (0 \leq t \leq \delta) \tag{17}$$

Assume for the moment that

$$F_1(0) = y_0^*(Ay_0) \neq 0 \quad \text{and} \quad F_2(0) = B^* y_0^* \neq 0. \tag{18}$$

By differentiating in $t = 0$ and using Lemma 1.1 for the derivative of the function $t \to \|F_2(t)\|$, we get from (17) that

$$\text{Re } \left(\frac{|F_1(0)|}{F_1(0)} F_1'(0) \right) + \tau \left(\frac{F_2(0)}{\|F_2(0)\|}\ ,\ F_2'(0) \right) \leq 0, \tag{19}$$

where the function τ must be calculated in l_1.

Recall that all the above arguments depend on a fixed vector $e \in Y$ which only satisfies (8). Let us see what is the effect of replacing $-e$ for e. In view of (13), k_0 (hence also k_0^*) changes its sign, so (15) tells us that also $F_1'(0)$ changes its sign. Therefore, by summing up (19) and the corresponding inequality for $-e$ we get

$$\tau \left(\frac{B^* y_0^*}{\|B^* y_0^*\|}\ ,\ B^* W^* k_0^* \right) + \tau \left(\frac{B^* y_0^*}{\|B^* y_0^*\|}\ ,\ -B^* W^* k_0^* \right) = 0 \tag{20}$$

(actually we get only one inequality, but the subadditivity of τ in the second variable gives the other).

The last trick will be to take for e a suitable element in the unit vector basis of Y, that is, $e = e_n$ with $n \notin \Lambda$, so that the requirements (8) are fulfilled. Then $h = We_n = \rho_n h_n$, so (13) gives

$$k_0 = r^{-1}(\rho_n h_n - r^{-1} \rho_n^2 \text{ Re } y_0(n)\ h_0)$$

and

$$B^*W^*k_0^* = \frac{\rho_n^2}{r^2}(rB^*e_n^* - (\text{Re } y_0(n))B^*W^*h_0^*)$$

We now put this value of k_0^* in (20), divide by ρ_n^2 and let $n \to \infty$. Since $\{y_0(n)\} \to 0$ and τ is a continuous function in the second variable, we get

$$\lim_{n\to\infty}\left[\tau\left(\frac{B^*y_0^*}{\|B^*y_0^*\|}\ ,\ B^*e_n^*\right)+\tau\left(\frac{B^*y_0^*}{\|B^*y_0^*\|}\ ,\ -B^*e_n^*\right)\right]=0 \tag{21}$$

This is the point where Fact 2.2 will play its important role. Since the sequence $B^*y_0^*$ has only finitely many nonzero terms, we can find a $z \in S_{c_0}$ such that $[B^*y_0^*](z) = \|B^*y_0^*\|$ and $z(n) = 0$, provided that n is large enough. For any scalar μ with $|\mu| = 1$ we have that $z \pm \mu e_n \in D\left(l_1, \frac{B^*y_0^*}{\|B^*y_0^*\|}\right)$ (consider $z \pm e_n$ as elements of $l_\infty \cong l_1^*$), so

$$\text{Re}[B^*e_n^*](z+\mu e_n) \le \tau\left(\frac{B^*y_0^*}{\|B^*y_0^*\|}\ ,\ B^*e_n^*\right)$$

and

$$-\text{Re}[B^*e_n^*](z-\mu e_n) \le \tau\left(\frac{B^*y_0^*}{\|B^*y_0^*\|}\ ,\ -B^*e_n^*\right).$$

It follows from (21) that

$$\lim_{n\to\infty} \text{Re } [B^*e_n^*](\mu e_n) = 0$$

so we have finally shown that $\{e_n^*(Be_n)\} \to 0$, and it is now pretty clear that B can not be close to the identity operator from Y onto c_0.

The above arguments are valid under the assumptions (18) that $y_0^*(Ay_0) \neq 0$ and $B^*y_0^* \neq 0$. The first one is easy to remove. If $y_0^*(Ay_0) = 0$, instead of (17) we have something better,

$$\|F_2(t)\| \le |F_1(t)| + \|F_2(t)\| \le \|F_2(0)\|,$$

so we simply forget about the function F_1 and differentiation in the above inequality leads directly to (20). On the other hand, the assumption $B^*y_0^* \neq 0$ is crucial to our argument. To be sure that this condition is satisfied we simply impose an additional restriction on the operator T. By Fact 2.3, if $B^*y_0^* = 0$, we have

$$\|B^*y^*\| \le |y^*(Ay)| + \|B^*y^*\| \le |y_0^*(Ay_0)| \le \|A\|,$$

and this holds for any norm attaining functional $y^* \in S_{Y^*}$. By the Bishop-Phelps Theorem [6] this implies $\|B\| \le \|A\|$. Therefore, we have proved the following.

PROPOSITION 2.4: *Let Y, X be the Banach spaces defined in 2.1, let $A \in L(Y)$, $B \in L(c_0, Y)$ satisfy $\|A\| < \|B\|$ and consider the operator $T \in L(X)$ defined by*

$$T(y+z) = Ay + Bz \quad (y \in Y \ , \ z \in c_0)$$

If T attains its numerical radius, then

$$\lim_{n\to\infty} e_n^*(Be_n) = 0$$

where $\{e_n\}$ and $\{e_n^*\}$ denote the unit vector bases of c_0 and l_1, respectively.

Our main result follows easily from the above proposition.

THEOREM 2.5: *Let X be the Banach space defined in 2.1. Then the set of numerical radius attaining operators is not dense in $L(X)$ for the norm topology.*

Proof: Consider the operator $S \in L(X)$ given by

$$S(y+z) = I(z) \quad (y \in Y \ , \ z \in c_0)$$

where $I \in L(c_0, Y)$ is the identity operator. If P and Q are the projections from X onto Y and c_0 respectively, we have clearly $S = PS$. We prove that S can not be approximated in norm by elements in $R(X)$.

Assume, on the contrary, that $T_k \in R(X)$ for all k and that the sequence $\{T_k\}$ converges in norm to S. Then $\{PT_k\}$ also converges to S and $\{QT_k\} \to 0$, hence $\{v(PT_k)\} \to v(S) > 0$ and $\{v(QT_k)\} \to 0$. It follows that $v(PT_k) > v(QT_k)$ for large enough k and, by Lemma 1.2, we have that $PT_k \in R(X)$, also for large enough k. Thus we can assume that $PT_k = T_k$, so that T_k has the form

$$T_k(y+z) = A_k y + B_k z \quad (y \in Y \ , \ z \in c_0)$$

where $A_k \in L(Y)$, $B_k \in L(c_0, Y)$, for all k. The definition of S implies that $\{A_k\} \to 0$, while $\{B_k\} \to I$, so $\|A_k\| < \|B_k\|$ for $k > k_0$ (say). From the above proposition we obtain that

$$\lim_{n\to\infty} e_n^*(B_k e_n) = 0 \quad (k > k_0).$$

Then from

$$1 = e_n^*(Ie_n) \le |e_n^*(B_k e_n)| + \|B_k - I\|,$$

by letting $n \to \infty$ we deduce that $\|B_k - I\| \geq 1$ for $k > k_0$, a contradiction. ∎

References

[1] M. Acosta, *Operadores que alcanzan su radio numérico,* Tesis Doctoral, Universidad de Granada, 1990.

[2] M. Acosta and R. Payá, *Denseness of operators whose second adjoints attain their numerical radii,* Proc. Amer. Math. Soc. **105** (1989), 97–101.

[3] M. Acosta and R. Payá, *Numerical radius attaining operators and the Radon-Nikodym property,* to appear in Bull. London Math. Soc.

[4] F. Bauer, *On the field of values subordinate to a matrix,* Numer. Math. **4** (1962), 103–111.

[5] I. Berg and B. Sims, *Denseness of numerical radius attaining operators,* J. Austral. Math. Soc. **36** (1984), 130–133.

[6] E. Bishop and R. Phelps, *A proof that every Banach space is subreflexive,* Bull. Amer. Math. Soc. **67** (1961), 97–98.

[7] F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras,* London Math. Soc. Lecture Note Series, Vol. 2, Cambridge Univ. Press, 1971.

[8] F. Bonsall and J. Duncan, *Numerical ranges II,* London Math. Soc. Lecture Note Series, Vol. 10, Cambridge Univ. Press, 1973.

[9] F. Bonsall and J. Duncan, *Numerical ranges,* in: *Studies in Functional Analysis* (R. Bartle ed.), pp. 1–49, MAA Studies in Math., Vol. 21, 1980.

[10] C. Cardassi, *Numerical radius attaining operators on $C(K)$,* Proc. Amer. Math. Soc. **95** (1985), 537–543.

[11] C. Cardassi, *Numerical radius attaining operators,* in *Banach Spaces. Proceedings Missouri 1984* (N. Kalton and E. Saab, eds.), pp. 11–14, Lecture Notes in Math., Vol. 1166, Springer-Verlag, Berlin, 1985.

[12] C. Cardassi, *Density of numerical radius attaining operators on some reflexive spaces,* Bull. Austral. Math. Soc. **31** (1985), 1–3.

[13] C. Cardassi, *Numerical radius attaining operators on $L_1(\mu)$,* preprint.

[14] N. Dunford and J. Schwartz, *Linear Operators. Part I,* Interscience Publishers, New York, 1958.

[15] P. Halmos. *A Hilbert Space Problem Book,* Van Nostrand, New York, 1967.

[16] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148.

[17] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29–43.

[18] S. Mazur, *Uber konvexe Mengen in linearen normierten Raumen*, Studia Math. **4** (1933), 70–84.

[19] J. Mena, R. Payá and A. Rodríguez, *Semisummands and semiideals in Banach spaces*, Israel J. Math. **51** (1985), 33–67.

[20] B. Sims, *On numerical range and its applications to Banach algebras,* Ph.D. Dissertation, University of Newcastle, Australia, 1972.

[21] O. Toeplitz, *Das algebraische Analogon zu einem Satze von Fejer*, Math. Z. **2** (1918), 187–197.